Wilberforce's pendulum revisited

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If its radius of gyration is properly tuned, a mass attached to a simple helical spring is able to perform longitudinal and torsional oscillations that are coupled with one another and exhibit the beats characteristic to coupled oscillations. The mechanism of coupling as well as the corresponding differential equations are derived. For the case of resonance, the solutions are compared with experimental results.

I. INTRODUCTION

In 1985 R. L. Wilberforce, a demonstrator of physics at the renowned Cavedish Laboratory, showed that a helical spring with an attached cylindrical mass as shown in Fig. 1 is able to perform longitudinal and torsional oscillations. They are coupled with one another and exhibit the characteristic beats, when the moment of inertia of the cylinder is properly tuned by means of the four metal knurled nuts screwed onto threaded rods. Today, Wilberforce's pendulum belongs to the classics of classroom demonstrations, and many physics textbooks mention this amazing phenomenon. Nevertheless, a detailed discussion is not easily found in the literature, since the original paper by Wilberforce is based upon two formulas describing the coupling of longitudinal and torsional motions that have been transferred from a textbook whose bibliographical data are not given. A second paper by Arnold Sommerfeld gives an exhaustive theory together with experiments and an accurate measurement of Poisson's ratio, but unfortunately it was published in a German "Festschrift," which is not available in most libraries. Forty years later, when Sommerfeld published his famous lectures on theoretical physics, he only included a rather short derivation using Castiglione's principle. It states that all statically indeterminate systems in equilibrium have minimum energy of deformation, but of course it is not able to tell us the mechanism or the reason for the coupling. This article is intended to address the deficiency, as noted by Jearl Walker, that this pendulum "seems to have been neglected over the years."4

II. STRETCHING AND ROTATING A HELICAL SPRING

Let us start with the discussion of a simple helical spring, the n windings of which may nearly touch each other in the unloaded state. That spring consists of 2n half-turns with negligible pitch. The deformations of each half-turn can be easily calculated. Since all stresses are constant over the length of such a spring, the deformations sum up along all half-windings; thus the longitudinal and rotational excursions of the mass attached to the spring can be calculated. In the following the symbols of the physical quantities describing a half-turn of the spring are printed with a bar; without bar they belong to the whole spring.

As usual, the torque $\mathbf{M}$ acting on the half-turn is to be resolved in components $[M_x, M_y, M_z]$ parallel to the orthonormal unit vectors $[\mathbf{u}, \mathbf{n}, \mathbf{b}]$ shown in Fig. 2. The first torque $M = F D_m/2 FR_m$ twists the wire of the spring by

Fig. 1. Wilberforce pendulum manufactured by Leybold-Heraeus GmbH. In the picture most of the spring is disabled by the clamp. This does not affect the resonance condition for the coupled oscillations. Technical data: mass of spring $m_s = 77$ g; mass of pendulum $m = 495$ g; length of spring $L_0 = 1765$ mm; number of turns $n = 125$; pitch angle $\alpha = 0.143$ rad; mean diameter of helix $D_m = 30.5$ mm; diameter of wire $d = 1.4$ mm; restoring force constant $D = 3.2$ N/m; restoring torque $D^* = 0.88$ N mm/rad.
the angle \( \bar{\gamma} \), displaces the straining point by \( \bar{z} \), and generates the helix angle \( \bar{a} = \alpha \). From Fig. 2 follows that \( z = R_m \bar{\gamma} = \alpha \pi R_m \). By neglecting the insignificant influence of the curvature of the wire on the distribution of torsional stress, one can use the well-known formula for a twisted rod with length \( L \) and diameter \( d \):

\[
\bar{\gamma} = \left( \frac{L}{GJ_p} \right) M, \quad \bar{z} = \left( \frac{LR^2}{2GJ_p} \right) F
\] (1)

\( J_p \) is the polar geometrical moment of inertia with \( J_p = \pi d^4/32 \) in the case of circular cross section, and \( G \) is the modulus of transverse elasticity in shear. Since \( L/L = \gamma/\bar{\gamma} = z/\bar{z} = \alpha^2 \), the bars in (1) may be omitted, yielding the restoring force constant of the whole helical spring with \( n \) turns:

\[
D = \frac{GJ_p}{LR^2} = \frac{Gd^4}{8nD_m^3},
\] (2)

with its characteristic fourth power of the diameter of the wire.

In differential geometry \( \tau = \cos \alpha \sin \alpha / R_m \approx \alpha / R_m \) stands for the torsion of a helical line with small pitch \( \alpha < 1 \), describing its tendency to leave the \((x,y)\) plane. With (1) it can be written as

\[
\tau = \frac{\gamma}{L} = \frac{\gamma}{L} = \frac{M_t}{GJ_p}.
\] (3)

Helical springs are often coiled up with an intrinsic pitch showing a geometric torsion \( \tau \neq 0 \) even in the absence of any torque. Then, the torque \( M_t \) generates an additional torsion \( \Delta \tau = \Delta (\gamma / L) \). Because \( M_t \) as well as the bending torque \( M_b \) (discussed below) cannot influence the length \( L \) of the wire of a spring, from \( \Delta L = 0 \) results

\[
\Delta \tau = \Delta \left( \frac{\gamma}{L} \right) = \Delta \left( \frac{\gamma}{L} \right) = \frac{z}{R_m} = \frac{M_t}{GJ_p}.
\] (4)

Let us now turn our attention to \( M_b \), the second component of the torque that bends the wire of our half-turn. Physics textbooks give the formula describing the curvature of a bent beam under the influence of a bending moment \( \kappa_b = 1/R_m = M_b/(EJ_a) \) with Young's modulus \( E \) and the axial geometrical moment of inertia \( J_a \) \( (J_a = \pi R_m^4/2) \) in the case of a circular cross section. Since our half-turn without any stress already exhibits an angle \( \phi_0 = \pi \) and a curvature \( \kappa_b = 1/R_m \), the bending moment again generates an additional curvature

\[
\Delta \kappa_b = \Delta \left( \frac{1}{R_m} \right) = \frac{M_b}{EJ_a} = \frac{1}{\pi} \frac{1}{R_m^2} \Delta R_m.
\] (5a)

Taking \( \Delta L = \Delta (\phi_0 R_m) = 0 = \Delta \phi R_m + \pi \Delta R_m \), we get

\[
\Delta \phi = \frac{M_b}{EJ_a} = \frac{\phi}{\pi R_m} = \frac{\phi}{L}.
\] (5b)

Again, the bars in this formula can be omitted, thus yielding the expressions for the angle of rotation, \( \phi \), and the restoring torque, \( D^* \), of the whole spring:

\[
\phi = \Delta \kappa_b, \quad L = (L/EJ_a) M_b = (2\pi R_m n/EJ_a) M_b,
\] (6)

\[
D^* = \frac{M_b}{\phi} = \frac{EJ_a}{L} = \frac{Ed^4}{64D_m n}.
\] (7)

The third component \( M_a \) generates a curvature \( \Delta \kappa_a \) that primarily influences the pitch of a helical spring, but not its radius of curvature, \( R_m \). Consequently, \( M_a \) is of no importance to the Wilberforce pendulum and need not be discussed here.

Now, we have to consider helical springs with pitch angles different from zero. In Wilberforce's pendulum such a pitch results from balancing out gravity acting on the pen-
The drawing is for \( n = 1 \). (b) Unwound half-turn. If the mass of the pendulum is rotated with constant \( z = 0 \), \( \alpha_0 \) and \( \gamma_0 \) are not altered in contrast to \( n, R_m \), and \( \kappa_0 \).

This generates a torque of reaction \( \vec{M}_s \). With the help of (5) and (7), one gets

\[
\vec{M}_s = \Delta \kappa_b \, EJ_s = \left( \frac{EJ_s}{R_m \, L} \right) \alpha_0 \varepsilon = \left( D \ast R_m \right) \alpha_0 \varepsilon \approx \vec{M}_s,
\]

where \( \vec{M}_s \) is parallel to the vector \(-b\).

In our case of small pitch angle \( \alpha_0 \), \( \vec{M}_s \) is almost parallel to the \( z \) axis, and its direction depends on the sign of torsion. Given freedom to turn, a stretching of the spring by \( z \) gives a rotation of

\[
\phi = - \frac{\vec{M}_s}{D^*} = - \left( \frac{\alpha_0}{R_m} \right) z.
\]

Let us now consider the consequences of a torque \( \vec{M}_s \approx \vec{M}_b \), rotating the spring about its longitudinal axis (\( z \) axis). Surely, a change \( \Delta \kappa_b \) will result, but what about the torsion? To answer this question one may refer to Fig. 3(a), illustrating an unwound helical spring. In this case a thin thread fixed between the mass of the pendulum and the floor would make \( z = \Delta (nH) = 0 \). Because \( \Delta L = 0 \), then the pitch angle \( \alpha_0 \) and \( nR_m \) are not affected by the rotation. Moreover, from the picture of the unwound half-turn [Fig. 3(b)], it can be seen that by increasing \( n \) the angle of torsion \( \gamma_0 \) of a half-turn remains unaffected, whereas the torsion angle \( \gamma_0 = 2n\gamma_0 \) of the whole spring increases by

\[
\Delta \gamma_0 = \gamma = 2n\gamma_0 \Delta n = 2n\gamma_0 (\phi/2\pi) = \alpha_0 \phi.
\]

From this, together with (4) and (6), follows

\[
\Delta \tau = \gamma/L = \alpha_0 (\phi/L) = \alpha_0 \Delta \kappa_b.
\]

Hence a force

\[
\vec{F}_s = \frac{GJ_s}{R_m} \Delta \tau = \frac{GJ_s}{R_m} \alpha_0 \phi = D\alpha_0 R_m \phi
\]

is generated, which tends to reduce the length of the spring. If the straining point is not fixed, it will be displaced by

\[
z = - \frac{\vec{F}_s}{D} = - \alpha_0 R_m \phi.
\]

Thus a helical spring contracts (\( z < 0 \)) if it is coiled (\( \Delta n > 0 \) and \( \phi > 0 \)) by a torque \( \vec{M}_s \). On the other hand, \( L_0 \) increases (\( z > 0 \)) if \( n \) is reduced (\( \phi < 0 \)).

If the mass of the pendulum is vertically displaced by \( z = F/D \), while some frictionless guiding device prevents it from rotating, then Eqs. (5) and (8) yield an increase in curvature given by

\[
\Delta \kappa_b = \Delta (1/R_m) = \alpha_0 \varepsilon / R_m L.
\]
III. DIFFERENTIAL EQUATIONS OF COUPLED OSCILLATIONS

If the mass \( m \) with its equation of inertia \( J \) is simultaneously displaced by \( z \) and rotated by \( \phi \), an acceleration

\[
\ddot{z} = (\ddot{F}_x + \ddot{F}_\phi)/m = -(D/m)(z + \alpha_0 R_m \phi), \tag{16}
\]

and an angular acceleration

\[
\ddot{\phi} = \frac{\ddot{M}_\phi + \ddot{M}_z}{J} = -\frac{D^*}{J}(\phi + \frac{\alpha_0}{R_m} z) \tag{17}
\]

results. These equations can be written in a form well known from coupled pendulums:

\[
\ddot{z} + \frac{(D/m)(z + k_1 \phi)}{J}, \quad \ddot{\phi} + \frac{D^*/J}{J}(\phi + k_2 z) = 0. \tag{18}
\]

Here \( k_1 = \alpha_0 R_m \) and \( k_2 = \alpha_0/R_m \) are the small coupling constants. For simplicity, let us look at the solution in case of resonance:

\[
\omega_0^2 = \frac{D^*}{m} = \frac{GL_p}{R_m L_m} = \frac{EJ_o}{LJ} = \frac{EJ_o}{LmR^2} \tag{19}
\]

\((R^2 = J/m)\) is the square of the radius of gyration of the suspended mass). This case can easily be investigated by properly tuning \( R \) to the elastic constants \( E \) and \( G \) and Poisson’s ratio \( \mu \):

\[
R^2/R_m = E/2G = 1 + \mu. \tag{20}
\]

If we put

\[
z_0 = z_0 \sin \omega t, \quad z_{11} = z_{00} \sin \omega t, \quad \phi_1 = \phi_0 \sin \omega t, \quad \phi_{11} = -\phi_0 \sin \omega t, \tag{21}
\]

into Eqs. (18) and substitute \( \omega_0^2 = D/m \), we get

\[
(\pm \omega^2 \pm \omega_0^2) z_0 + \omega_0^2 k_1 \phi_0 = 0, \tag{22}
\]

\[
(\pm \omega^2 \pm \omega_0^2) \phi_0 + \omega_0^2 k_2 z_0 = 0,
\]

from which the frequencies of the two fundamental modes can be calculated by eliminating \( z_0/\phi_0 \):

\[
\omega_{11} = \omega_0 \sqrt{1 \pm \sqrt{k_1^2 k_2^2}} \Rightarrow \omega_{11} \approx \omega_0 (1 \pm \alpha_0/2). \tag{23}
\]

The upper sign stands for in-phase oscillations \( \phi(t) \) and \( z(t) \); the lower one must be used with opposite phase.

As is the case with common coupled pendulums, the geometric mean of the coupling constants yields the difference in frequency between both fundamental modes. But in contrast, the in-phase oscillation of the Wilberforce pendulum has the higher frequency since the restoring forces \( \ddot{F}_x \) and \( \ddot{F}_\phi \) are added as are the torques \( \ddot{M}_\phi \) and \( \ddot{M}_z \).

As a consequence of (22), pure fundamental modes can only be observed for very special initial values \( z(t = 0) = z_0 \) and \( \phi(t = 0) = \phi_0 \). In most cases where both fundamental oscillations are superimposed, characteristic beats result with a beating frequency

\[
\omega_s = 2\pi/T_s = \omega_1 - \omega_{11} = \omega_0 \alpha_0 = \left[ (\omega_1 + \omega_{11}) / 2 \right] \alpha_0. \tag{24}
\]

The pronounced 100% modulation results from a \( \pi/2 \)-phase difference, e.g., from \( z(t = 0) = z_0 \) and \( \phi_0 = 0 \). Then, the entire energy of oscillation is transferred from translational to rotational energy and vice versa.

IV. EXPERIMENTS WITH A WILBERFORCE PENDULUM

Experiments were performed with the pendulum shown in Fig. 1 and a homemade spring with a somewhat larger pitch. The caption of Fig. 1 summarizes the technical data. In a first experiment the restoring force \( D \) as well as the restoring torque \( D^* \) were determined by applying a static external torque \( M \) or (with higher precision) by a static load \( F \). They were also calculated from measurements of the period of oscillation where mass and moment of inertia of the spring itself were taken into account. Moreover, additional masses were fixed to the threaded rods to remove resonance and to prevent any noticeable beating. The latter \( D \) and \( D^* \) values were approximately 1% higher when thin threads fixed to the pendulum’s mass suppressed either rotational or translational movements. The elastic constants \( E, G, \) and \( \mu \) of the wire are not given in the table, but can easily be calculated using (2), (7), and (20).

Changing the static loads applied to the freely movable mass, one can simultaneously monitor \( z \) and \( \phi \), thus verifying (11) and (15). The measured values \( \phi/z = -7 \text{ rad/m} \) and \( \phi^2/\phi = 0.0023 \text{ m/rad} \) correspond well with the calculated ones. Next, the pendulum was tuned by screwing the four metal disks (shown in Fig. 1) in steps outside onto the threaded rods and measuring the beat period of the oscillations excited with \( z(t = 0) = z_0 \) and \( \phi_0 = 0 \). If the resonance condition (20) is fulfilled, then pronounced maxima in the beat period and beat modulation depth appear (Fig. 4). With the pendulum thus tuned, one can demonstrate experimentally the rather astonishing fact that neither length or diameter of the wire nor pitch and number of turns influences the resonance condition. This follows from (19) and (20).

The validity of Eq. (23) was tested in a further experiment by exciting the two fundamental modes of oscillation selectively. To achieve this one can stretch the spring in a series of measurements by a constant \( z_0 \) while incrementally increasing \( \pm \phi_0 \). If condition (22) is fulfilled, the pendulum oscillates for more than 50 periods without any visible beating (Fig. 5(a)). The Leybold pendulum demonstrates in the symmetric mode (\( \text{sgn } \phi = \text{sgn } z \)) \( \omega_{11} = 2.572 \text{ s}^{-1} \) \( (T_{11} = 2.443 \text{ s}) \) and with (\( \text{sgn } \phi = -\text{sgn } z \)) \( \omega_{11} = 2.304 \text{ s}^{-1} \) \( (T_{11} = 2.727 \text{ s}) \). From this one can calculate a beat period \( T_s = 23.4 \text{ s} \), which corresponds with the
As a consequence, the in-phase fundamental mode has to oscillate with the lower frequency. It seems that Sommerfeld did not mention in his experiments that this is not true, that the in-phase oscillation in fact has the higher frequency!

Of course, Eqs. (25) and (26) are right. Moreover, the minus sign in (26) is reasonable, since by increasing $z$ the wire of a spring in fact becomes more straight, thus diminishing its curvature $\kappa$. Perhaps Sommerfeld omitted that two $z$-dependent terms contribute to the change in curvature of a helix:

$$ \Delta \kappa = \Delta (\cos^2 \alpha_0/R_m) = \frac{\partial \kappa}{\partial \alpha_0} \Delta \alpha_0 + \frac{\partial \kappa}{\partial R_m} \Delta R_m$$

$$= -\frac{2\alpha_0}{R_m} z + \frac{\alpha_0}{R_m} z = \Delta \kappa_n + \Delta \kappa_s. \quad (28)$$

While the spring is strained, only $\Delta \kappa_s$ generates a torque that is able to couple longitudinal and rotational oscillations. Therefore, only the part of (28) with the “right,” i.e., the positive sign, becomes effective and has to be considered.

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5. Since the sign of torsion does not influence the mechanical properties of a spring, $r$ stands for $|r|$ in the following.