W15D1-1 Group Problem: Elliptic Orbits:
*The Motion of SO-2 around the Black Hole at the Galactic Center*

The UCLA Galactic Center Group, headed by Dr. Andrea Ghez, reported the following data (see [http://www.astro.ucla.edu/~jlu/gc/](http://www.astro.ucla.edu/~jlu/gc/) for information about the research group, and [http://www.astro.ucla.edu/~ghezgroup/gc/images/2006orbits_animfull.gif](http://www.astro.ucla.edu/~ghezgroup/gc/images/2006orbits_animfull.gif) for an animation of the orbits about the galactic center) for the orbits of eight stars within 0.8″ × 0.8″ of the galactic center.

![Figure 1: Stellar Orbits near Galactic Center](image)

The orbits of the stars are shown in Figure 1.
A standard astronomical unit is the parsec. One parsec is the distance at which there is one arcsecond = 1/3600 deg angular separation between two objects that are separated by the distance of one astronomical unit, \(1\text{AU} = 1.50\times10^{11} \text{m} \), which is the mean distance between the earth and the sun. One astronomical unit is roughly equivalent to eight light-minutes, \(1\text{AU} = 8.3 \text{l-min} \). One parsec is equal to 3.26 light-years, where one light-year is the distance that light travels in one earth year, \(1\text{pc} = 3.26\text{ly} = 2.06\times10^5 \text{AU} \), where \(1\text{ly} = 9.46\times10^{15} \text{m} \). (The unit “light-year” is now a “deprecated unit;” see [http://pdg.lbl.gov/2006/reviews/astrorpp.pdf](http://pdg.lbl.gov/2006/reviews/astrorpp.pdf) for the most recent values of astrophysical constants and parameters.) The orbital data for the stars S0-2, S0-16, and S0-19 are as follows\(^1\):

<table>
<thead>
<tr>
<th>Star</th>
<th>Period (yrs)</th>
<th>Eccentricity</th>
<th>Semimajor axis (10(^{-3})arc sec)</th>
<th>Periapse (AU)</th>
<th>Apoapse (AU)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S0-2</td>
<td>15.2 (0.68/0.76)</td>
<td>0.8763 (0.0063)</td>
<td>120.7 (4.5)</td>
<td>119.5 (3.9)</td>
<td>1812 (73)</td>
</tr>
<tr>
<td>S0-16</td>
<td>29.9 (6.8/13)</td>
<td>0.943 (0.019)</td>
<td>191 (24)</td>
<td>87 (17)</td>
<td>2970 (560)</td>
</tr>
<tr>
<td>S0-19</td>
<td>71 (35/11000)</td>
<td>0.889 (0.065)</td>
<td>340 (220)</td>
<td>301 (41)</td>
<td>5100 (3600)</td>
</tr>
</tbody>
</table>

The period of S0-2 satisfies Kepler’s Third Law, given by

\[
T^2 = \frac{4\pi^2a^3}{G(m_1 + m_2)}, \tag{1}
\]

where \(m_1\) is the mass of S0-2, \(m_2\) is the mass of the black hole, and \(a\) is the semimajor axis of the elliptic orbit of S0-2.

The orbit data is given in terms of properties of the elliptic orbit. Consider the ellipse shown in the figure below.

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In Figure 2, let $a$ denote the semimajor axis, $b$ denote the semiminor axis and $x_0$ denote the location of the center of the ellipse from one focal point $P$.

The orbit equation for the system is given by

$$r = \frac{r_0}{1 - \epsilon \cos \theta},$$

where $r_0$ and the eccentricity $\epsilon$ are two constant for a given orbit.

The semimajor axis $a$ is given by

$$a = \frac{r_p + r_a}{2}$$

where the distance of furthest approach is denoted by $r_a$, called the apoapse, and the distance of nearest approach is denoted by $r_p$, called the periapse.

Questions:

a) Using the results in the data table for the star S0-2, find the length of the semimajor axis. Give your answer in both astronomical units (AU) and meters.
b) Using the results in the data table for the star S0-2, find the mass of the black hole that the star S0-2 is orbiting. How many solar masses does this correspond to? The Universal Gravitation Constant is \( G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2\cdot \text{kg}^{-2} \) and the mass of the sun is \( m_{\text{sun}} = 1.99 \times 10^{30} \text{ kg} \).

c) Use the equations for constant energy and angular momentum to find the speed of S0-2 at periapse and apoapse.

**Solutions:**

a) From Equation (3), the semimajor axis is

\[
a = \frac{r_p + r_a}{2} = \frac{119.5 \text{ AU} + 1812 \text{ AU}}{2} = 965.8 \text{ AU}.
\]

In SI units (meters), this is

\[
a = 965.8 \text{ AU} \times \frac{1.50 \times 10^{11} \text{ m}}{1 \text{ AU}} = 1.45 \times 10^{14} \text{ m}.
\]

b) The mass \( m_1 \) of the star S0-2 is much less than the mass \( m_2 \) of the black hole, and Equation (1) may be expressed as

\[
T^2 = \frac{4\pi^2 a^3}{G m_2}.
\]

Solving for the mass \( m_2 \) and inserting the numerical values, and using \( 1 \text{ yr} = 3.16 \times 10^7 \text{ s} \),

\[
m_2 = \frac{4\pi^2 a^3}{G T^2} = \frac{(4\pi^2)(1.45 \times 10^{14} \text{ m})^3}{(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2\cdot \text{kg}^{-2})(15.2 \text{ yr})(3.16 \times 10^7 \text{ s} \cdot \text{yr}^{-1})^2}
\]

\[
= 7.79 \times 10^{34} \text{ kg}.
\]

In solar masses, this is
\[ \frac{m_2}{m_{\text{sun}}} = \frac{7.79 \times 10^{34} \text{ kg}}{1.99 \times 10^{30} \text{ kg}} = 3.91 \times 10^{6} . \]  
\[ (8) \]

c) Denote the speeds at periapse and apoapse as \( v_p \) and \( v_a \). Conservation of mechanical energy is then expressed as

\[ \frac{1}{2} \mu v_p^2 - G \frac{m_1 m_2}{r_p} = \frac{1}{2} \mu v_a^2 - G \frac{m_1 m_2}{r_a} . \]  
\[ (9) \]

In the limit \( m_2 \gg m_1 \) (the same limiting case used in part b)), \( \mu = m_1 \). Inserting this into Equation (9) and canceling a factor of \( m_1 \),

\[ \frac{1}{2} v_p^2 - G \frac{m_2}{r_p} = \frac{1}{2} v_a^2 - G \frac{m_2}{r_a} . \]  
\[ (10) \]

Rearranging to put the terms with the speeds and distances together,

\[ v_p^2 - v_a^2 = 2Gm_2 \left( \frac{1}{r_p} - \frac{1}{r_a} \right) . \]  
\[ (11) \]

Conservation of angular momentum relates the speeds and distances by

\[ r_p v_p = r_a v_a ; \]  
\[ (12) \]

solving for \( v_p \) in terms of \( v_a \) and substituting into Equation (11) gives

\[ v_p^2 \left( 1 - \frac{r_p^2}{r_a^2} \right) = 2Gm_2 \left( \frac{1}{r_p} - \frac{1}{r_a} \right) . \]  
\[ (13) \]

With a little algebra, Equation (13) is re-expressed as

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\[ v_p^2 \left( \frac{r_a^2 - r_p^2}{r_a^2} \right) = 2Gm_2 \left( \frac{r_a - r_p}{r_p r_a} \right) \]

\[ v_p^2 \left( \frac{r_a^2 + r_p^2}{r_a^2} \right) (r_a - r) = \frac{2Gm_2}{r_p r_a} (r_a - r_p) \]

Setting aside (for now) the \( r_p = r_a \) case of circular motion, canceling the factor of \( r_a - r_p \) in Equation (14) and solving for \( v_p \) gives

\[ v_p^2 = \frac{2Gm_2}{r_p r_a} \left( \frac{r_a^2}{r_a + r_p} \right) = \frac{2Gm_2}{r_p (2a)} \]

where \( a \) is the semimajor axis found in part a). Inserting numerical values,

\[ v_p = \sqrt{\frac{Gm_2 \cdot r_a}{a \cdot r_p}} = \sqrt{\left( \frac{6.67 \times 10^{-11} \text{N} \cdot \text{m}^2 \cdot \text{kg}^{-2}}{1.45 \times 10^{14} \text{m}} \right) \left( \frac{7.79 \times 10^{24} \text{kg}}{119.5} \right)} \]

\[ = \frac{7.38 \times 10^6 \text{m} \cdot \text{s}^{-1}}{} \]

The speed \( v_a \) at apoapse could be found by re-deriving Equation (13) in terms of \( v_a \) instead of \( v_p \), but it’s much simpler to use Equation (12);

\[ v_a = \frac{r_a}{v_p} = \left( \frac{7.38 \times 10^6 \text{m} \cdot \text{s}^{-1}}{} \right) \left( \frac{1812}{119.5} \right) = 4.87 \times 10^5 \text{m} \cdot \text{s}^{-1} \].

The case of a circular orbit, \( r_p = r_a = r_c \), with \( v_p = v_a = v_c \), is addressed by using the kinematics and dynamics of uniform circular motion,

\[ m_i \frac{v_c^2}{r_c} = G \frac{m_i m_2}{r_c^2} \]

\[ v_c^2 = \frac{Gm_2}{r_c} \]

which is Equation (15) with \( r_p = r_a = r_c = a \). This is as it should be; a circular orbit should certainly not be treated as a special case.
Note that the mass $m_1$ of S0-2 did not enter into our calculations, under the assumption that $m_2 \gg m_1$. If this were not the case, we would need to know something about the orbit of the black hole, which is not observable.