Kinetic Energy, Work, and Power

8.01
W06D1
Forms of Energy

- kinetic energy
- gravitational energy
- elastic energy
- thermal energy
- electrical energy
- chemical energy
- electromagnetic energy
- nuclear energy
- mass energy
Energy Transformations

• Falling water releases stored ‘gravitational potential energy’ turning into a ‘kinetic energy’ of motion.

• Human beings transform the stored chemical energy of food into catabolic energy

• Burning gasoline in car engines converts ‘chemical energy’ stored in the atomic bonds of the constituent atoms of gasoline into heat

• Stretching or compressing a spring stores ‘elastic potential energy’ that can be released as kinetic energy
Energy Conservation

• Energy is always conserved

\[ \sum_{i=1}^{N} \Delta E_i = \Delta E_1 + \Delta E_2 + \ldots = 0 \]

• It is converted from one form into another, from an “initial state” to a “final state”

\[ \Delta E \equiv E_{\text{final}} - E_{\text{initial}} \]

• Energy can also be transferred from a system to its surroundings

\[ \Delta E_{\text{system}} + \Delta E_{\text{surroundings}} = 0 \]
Kinetic Energy

• Scalar quantity (reference frame dependent)

\[ K = \frac{1}{2} mv^2 \geq 0 \]

• SI unit is joule:

\[ 1 \text{J} \equiv 1 \text{kg} \cdot \text{m}^2/\text{s}^2 \]

• Change in kinetic energy:

\[ \Delta K = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_0^2 = \frac{1}{2} m(v_{x,f}^2 + v_{y,f}^2 + v_{z,f}^2) - \frac{1}{2} m(v_{x,0}^2 + v_{y,0}^2 + v_{z,0}^2) \geq 0 \]
Concept Question

Compared to the amount of energy required to accelerate a car from rest to 10 mph (miles per hour), the amount of energy required to accelerate the same car from 10 mph to 20 mph is

(1) the same
(2) twice as much
(3) three times as much
(4) four times as much
Kinematics: Recall Integration Formula

The $x$-component of the acceleration of an object is the derivative of the $x$-component of the velocity

$$a_x \equiv \frac{dv_x}{dt}$$

Therefore the integral of $x$-component of the acceleration with respect to time, is the $x$-component of the velocity

$$\int_{t_0}^{t_f} a_x \, dt = \int_{t_0}^{t_f} \frac{dv_x}{dt} \, dt = \int_{v_{x,0}}^{v_{x,f}} dv_x = v_{x,f} - v_{x,0}$$
Kinematics: An Integral Theorem for One Dimensional Motion

The integral of $x$-component of the acceleration with respect to the displacement of an object, is given by

$$\int_{x_0}^{x_f} a_x \, dx = \int_{x_0}^{x_f} \frac{dv_x}{dt} \, dx = \int_{x_0}^{x_f} dv_x \, \frac{dx}{dt} = \int_{v_{x_0}}^{v_{x_f}} v_x \, dv_x$$

$$\int_{x_0}^{x_f} a_x \, dx = \int_{v_{x_0}}^{v_{x_f}} d \left( \frac{1}{2} v_x^2 \right) = \frac{1}{2} \left( v_{x_f}^2 - v_{x_0}^2 \right)$$

Multiply both sides by the mass of the object

$$\int_{x_0}^{x_f} ma_x \, dx = \int_{v_{x_0}}^{v_{x_f}} d \left( \frac{1}{2} m v_x^2 \right) = \frac{1}{2} mv_{x_f}^2 - \frac{1}{2} mv_{x_0}^2 = \Delta K$$
Newton’s Second Law: An Integral Theorem for One Dimensional Motion

Newton’s Second Law
If $F$ is not constant, then (in one dimension),

$$F_x = ma$$

Therefore the integral theorem coupled with Newton’s Second Law becomes

$$\int_{x_0}^{x_f} ma_x \, dx = \int_{x_0}^{x_f} F_x \, dx = \Delta K$$
Work Done by a Constant Force for One Dimensional Motion

Definition:

The work $W$ done by a constant force with an $x$-component, $F_x$, in displacing an object by $\Delta x$ is equal to the $x$-component of the force times the displacement:

$$W = F_x \Delta x$$
Concept Question

When a person walks, the force of friction between the floor and the person's feet accelerates the person forward. The floor does

(1) Positive work on the person
(2) Negative work on the person
(3) No work on the person
Work done by Non-Constant Force: One Dimensional Motion

(Infinitesimal) work is a scalar

\[ \Delta W_i = (F_x)_i \Delta x_i \]

Add up these scalar quantities to get the total work as area under graph of \( F_x \) vs \( x \):

\[ W = \sum_{i=1}^{i=N} \Delta W_i = \sum_{i=1}^{i=N} (F_x)_i \Delta x_i \]

As \( N \to \infty \) and \( |\Delta x_i| \to 0 \)

\[ W = \lim_{N \to \infty} \sum_{\Delta x_i \to 0}^{i=N} (F_x)_i \Delta x_i = \int_{x=x_0}^{x=x_f} F_x \, dx \]
Table Problem: Work Done by Gravity Near the Surface of the Earth

Consider an object of mass \( m \) near the surface of the earth falling directly towards the center of the earth. The gravitational force between the object and the earth is nearly constant. Suppose the object starts from an initial point that is a distance \( y_0 \) from the surface of the earth and moves to a final point a distance \( y_f \) from the surface of the earth.

How much work does the gravitational force do on the object as it falls?
Work-Kinetic Energy Theorem for One Dimensional Motion

Work-energy theorem is the statement that the work done by a force in displacing an object is equal to the change in kinetic energy of the object

\[ W = \Delta K \]
Concept Question

Consider two carts, of masses $m$ and $2m$, at rest on an air track. If you push one cart for 3 s and then the other for the same length of time, exerting equal force on each, the kinetic energy of the light cart is

(1) larger than
(2) equal to
(3) smaller than

the kinetic energy of the heavy car.
A particle starts from rest at $x = 0$ and moves to $x = L$ under the action of a variable force $F(x)$, which is shown in the figure. What is the particle's kinetic energy at $x=L/2$ and at $x=L$?

1. $(F_{\text{max}})(L/2), (F_{\text{max}})(L)$
2. $(F_{\text{max}})(L/4), 0$
3. $(F_{\text{max}})(L), 0$
4. $(F_{\text{max}})(L/4), (F_{\text{max}})(L/2)$
5. $(F_{\text{max}})(L/2), (F_{\text{max}})(L/4)$
Table Problem: Work Done by the Spring Force

Connect one end of a spring of length $l_0$ with spring constant $k$ to an object resting on a smooth table and fix the other end of the spring to a wall. Stretch the spring until it has length $l$ and release the object.

How much work does the spring do on the object as a function of $x = l - l_0$, the distance the spring has been stretched or compressed?
Power

• The average power of an applied force is the rate of doing work

\[ \bar{P} = \frac{\Delta W}{\Delta t} = \frac{F_{\text{applied,}x} \Delta x}{\Delta t} = F_{\text{applied,}x} \bar{v}_x \]

• SI units of power: Watts

\[ 1 \text{ W} \equiv 1 \text{ J/s} = 1 \text{ kg} \cdot \text{m}^2/\text{s}^3 \]

• Instantaneous power

\[ P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = F_{\text{applied,}x} \left( \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \right) = F_{\text{applied,}x} \bar{v}_x \]
Dot Product

A scalar quantity

Magnitude:

\[ \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta \]

The dot product can be positive, zero, or negative

Two types of projections: the dot product is the parallel component of one vector with respect to the second vector times the magnitude of the second vector

\[ \vec{A} \cdot \vec{B} = |\vec{A}| (\cos \theta) |\vec{B}| = A_\parallel |\vec{B}| \]

\[ \vec{A} \cdot \vec{B} = |\vec{A}| (\cos \theta) |\vec{B}| = |\vec{A}| B_\parallel \]
Dot Product Properties

\[ \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} \]

\[ c\vec{A} \cdot \vec{B} = c(\vec{A} \cdot \vec{B}) \]

\[ (\vec{A} + \vec{B}) \cdot \vec{C} = \vec{A} \cdot \vec{C} + \vec{B} \cdot \vec{C} \]
Dot Product in Cartesian Coordinates

With unit vectors \( \hat{i}, \hat{j} \) and \( \hat{k} \)

\[
\begin{align*}
\hat{i} \cdot \hat{i} &= \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \\
\hat{i} \cdot \hat{j} &= \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{k} = 0
\end{align*}
\]

Example:

\[
\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}, \quad \vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}
\]

\[\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z\]
Work Done by a Constant Force

Definition: Work

The work done by a constant force $\vec{F}$ on an object is equal to the component of the force in the direction of the displacement times the magnitude of the displacement:

$$W = \vec{F} \cdot \Delta \vec{r} = |\vec{F}| |\Delta \vec{r}| \cos \theta = |\vec{F}| \cos \theta |\Delta \vec{r}| = F_\parallel |\Delta \vec{r}|$$

Note that the component of the force in the direction of the displacement can be positive, zero, or negative so the work may be positive, zero, or negative.
Work as a Dot Product

Let the force exerted on an object be

\[ \vec{F} = F_x \hat{i} + F_y \hat{j} \]

\[ F_x = F \cos \beta \]
\[ F_y = F \sin \beta \]

Displacement: \( \Delta \vec{r} = \Delta x \hat{i} \)

\[ W = \vec{F} \cdot \Delta \vec{r} = F \Delta x \cos \beta \]
\[ = (F_x \hat{i} + F_y \hat{j}) \cdot (\Delta x \hat{i}) = F_x \Delta x \]
Concept Question: Work

A ball is given an initial horizontal velocity and allowed to fall under the influence of gravity, as shown below. The work done by the force of gravity on the ball is:

(1) positive
(2) zero
(3) negative
Concept Question: Work

A comet is speeding along a hyperbolic orbit toward the Sun. While the comet is moving away from the Sun, the work done by the Sun on the comet is:

(1) positive  
(2) zero  
(3) negative
Work Done Along an Arbitrary Path

\[ \Delta W_i = \mathbf{F}_i \cdot \Delta \mathbf{r}_i \]

\[ W = \lim_{N \to \infty} \sum_{i=1}^{i=N} \mathbf{F}_i \cdot \Delta \mathbf{r}_i = \int_0^f \mathbf{F} \cdot d\mathbf{r} \]
Work in Three Dimensions

Let the force acting on an object be given by

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

The displacement vector for an infinitesimal displacement is

$$d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k}$$

The work done by the force for this infinitesimal displacement is

$$dW = \vec{F} \cdot d\vec{r} = (F_x \hat{i} + F_y \hat{j} + F_z \hat{k}) \cdot (dx \hat{i} + dy \hat{j} + dy \hat{k})$$

$$dW = F_x dx + F_y dy + F_z dz$$

Integrate to find the total work

$$W = \int_0^f \vec{F} \cdot d\vec{r} = \int_0^f F_x dx + \int_0^f F_y dy + \int_0^f F_z dz$$
Table Problem: Work Done by by the Inverse Square Gravitational Force

Consider an object of mass $m$ moving towards the sun (mass $m_s$). Initially the object is at a distance $r_0$ from the center of the sun. The object moves to a final distance $r_f$ from the center of the sun.

How much work does the gravitational force between the sun and the object do on the object during this motion?
Work-Energy Theorem in Three-Dimensions

Newton’s Second Law:

\[ F_x = ma_x , \quad F_y = ma_y , \quad F_z = ma_z \]

Total work:

\[
W = \int_{\vec{r}=\vec{r}_0}^{\vec{r}=\vec{r}_f} \vec{F} \cdot d\vec{r} = \int_{\vec{r}=\vec{r}_0}^{\vec{r}=\vec{r}_f} F_x \, dx + \int_{\vec{r}=\vec{r}_0}^{\vec{r}=\vec{r}_f} F_y \, dy + \int_{\vec{r}=\vec{r}_0}^{\vec{r}=\vec{r}_f} F_z \, dz
\]

becomes

\[
W = \int_{\vec{r}=\vec{r}_0}^{\vec{r}=\vec{r}_f} ma_x \, dx + \int_{\vec{r}=\vec{r}_0}^{\vec{r}=\vec{r}_f} ma_y \, dy + \int_{\vec{r}=\vec{r}_0}^{\vec{r}=\vec{r}_f} ma_z \, dz
\]
Work-Energy Theorem in Three-Dimensions

Recall

\[ \int_{x_0}^{x_f} ma_x \, dx = \int_{x_0}^{x_f} m \frac{dv_x}{dt} \, dx = \int_{x_0}^{x_f} m \frac{dx}{dt} \, dv_x = \int_{v_{x,0}}^{v_{x,f}} m v_x \, dv_x = \frac{1}{2} mv_{x,f}^2 - \frac{1}{2} mv_{x,0}^2 \]

Repeat argument for \( y \)- and \( z \)-direction

\[ \int_{y_0}^{y_f} ma_y \, dx = \frac{1}{2} mv_{y,f}^2 - \frac{1}{2} mv_{y,0}^2 \quad \int_{z_0}^{z_f} ma_z \, dx = \frac{1}{2} mv_{z,f}^2 - \frac{1}{2} mv_{z,0}^2 \]

Adding these three results

\[ W = \int_{z_0}^{z_f} (ma_x \, dx + ma_y \, dy + ma_z \, dz) \, dx \]

\[ W = \frac{1}{2} m (v_{x,f}^2 + v_{y,f}^2 + v_{z,f}^2) - \frac{1}{2} m (v_{x,0}^2 + v_{y,0}^2 + v_{z,0}^2) = \frac{1}{2} mv_{f}^2 - \frac{1}{2} mv_{0}^2 \]

\[ W = \Delta K \]