Johnson Noise and Shot Noise

Rachel L. Finck*
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In this experiment, we use Johnson noise and shot noise to measure several fundamental constants. Measurements of the resistance dependence of Johnson noise yield a value of $(2.55 \pm 0.03) \times 10^{-23} \text{J/K}$ for Boltzmann's constant, $k$. Temperature dependence gives a more accurate value of $k = (1.04 \pm 0.08) \times 10^{-23} \text{J/K}$, and also provides a determination of absolute zero as $(-310 \pm 30) \text{C}$. Initial measurements made of shot noise data show that the charge of an electron, $e$, is $(5.89 \pm 0.12) \times 10^{-20} \text{C}$, which is then estimated at $(2.36 \pm 0.05) \times 10^{-19} \text{C}$ after a correction for a systematic error.

1. BACKGROUND AND CONTEXT

Despite the wide acceptance of the atomic theory of matter, direct experimental evidence of atoms had not yet been measured at the turn of the twentieth century. The smallest amounts of matter that could possibly be observed still contained too many atoms to witness their "granularity." Although the equation of state of a mole of gas of pressure $P$, volume $V$, and temperature $T$ was known to be

$$PV = R_g kT,$$

and although the value of the constant $R_g$ had been measured, it was unknown how to measure its components, either the number of atoms in a mole, $N_A$, or the constant $k$ that relates the two quantities, such that $R_g = \frac{k}{N_A}$. Then, in 1910, Millikan’s oil drop experiment measured $e$, the charge of an electron, which further enabled precise determinations of numerous unknown constants that were related to $e$, including Avogadro’s number, $N_{AV}$, and Boltzmann’s constant, $k$.

It was twenty years later that a new way to measure these quantities was discovered by Johnson. He found the quantity of mean square noise in electrical systems to be analogous to gas pressure. Furthermore, through an analysis of the number of degrees of freedom of electrical oscillations in a transmission line, Nyquist was able derive an expression for the mean square "Johnson noise" with $k$ appearing as the only atomic constant. It was therefore possible to measure $k$ by measuring the Johnson noise, as we wish to do in this experiment. Additionally, the temperature dependence of the Johnson noise allows one to check the value of absolute zero on the Celcius temperature scale.

Millikan's discovery of the charge of an electron also had implications for another type of noise. The consistency of current as the flow of discrete charge carriers, rather than a mathematically continuous stream, will result in some inherent noise. It is this that is referred to as shot noise, first named so by Schottky in 1919, and its dependence on the charge a single electron provides another method by which to measure $e$, as we do in this experiment.

2. ESSENTIAL THEORY

2.1. Johnson Noise

Let $g$ be the number of accessible states of a system with fixed total energy $U$ and number of particles $N$. Then the fundamental entropy is defined as $\sigma = \ln g$. Suppose $U$ is increased by an amount $\Delta U$ while holding $N$ and the volume $V$ constant. Then the number of accessible states increases by an amount $\Delta \sigma$, leading to the definition of the fundamental temperature as

$$\frac{1}{\tau} = \left(\frac{\partial \sigma}{\partial U}\right)_{N,V}$$

From this definition, it is clear that $\tau$ is measured in the same units as energy. The familiar Kelvin temperature scale $T$, however, is defined as setting $0K$ to absolute zero and $273.16K$ to the triple point of water. These two scales are then linked by a proportionality constant $k$, where

$$\tau = kT$$

The constant $k$, known as Boltzmann’s constant, comes into play through a quantum statistical analysis of confined particles at thermal equilibrium at temperature $T$. In this system, the mean energy per translational degree of freedom is $NkT/2$ [1].

Johnson found an analogous electrical system. In a conductor, the electrical modes of oscillation are coupled to the thermal environment by charge carriers, and when thermally agitated cause "noise" across the conductor, which can be measured as a mean square voltage. This was shown by Nyquist to be measurable without $N$ or $e$ coming into the expression, allowing for $k$ to be measured directly.

Nyquist used the second law of thermodynamics and the equipartition theorem to derive the differential contribution to the mean square voltage across a resistor $R$ in the frequency interval $df$ to be

$$dV^2 = 4kTdf$$
It is important to note that this expression is valid only in the frequency range such that \( h \nu < kT \), in which the approximation that the mean energy of each oscillation mode is \( kT \) is valid [2].

When measuring the Johnson noise, it is desirable to reduce the RMS voltage by connecting the resistor to cables with a capacitance \( C \) that short circuits part of the signal. This changes the differential contribution to the RMS voltage to

\[
\frac{dV_J^2}{I} = 4Rf \frac{df}{kT}
\]

where

\[
R_f = \frac{R}{1 + (2\pi f CR)^2}
\]

With a frequency-dependent gain \( g(f) = V_0(f)/V_i(f) \), the differential contribution of Johnson noise to the voltage becomes

\[
dV_J^2 = |g(f)|^2 dV_J^2,
\]

and therefore, by Equations 5 and 7,

\[
V_J^2 = 4RkTG
\]

where

\[
G = \int_0^\infty \frac{|g(f)|^2 df}{1 + (2\pi f CR)^2}
\]

The integration in Equation 9 comes from estimating the noise across the resistor as a sum of Fourier components, the square of which equals the sum of the squares of the components by their orthogonality. Equation 8 therefore gives us both the resistance and temperature dependencies we require to measure both \( k \) and absolute zero on the Kelvin scale.

2.2. Shot Noise

Shot noise is defined as the fluctuations about an average value of current inherent in a source in which the passage of charge carriers are statistically independent events. It therefore follows that the magnitude of such fluctuations depends on the charge of each carrier. In this experiment, we use this principle to determine \( e \), the charge of an electron, by measuring the shot noise in a circuit.

Illumination of a photodiode causes electrons to be ejected from the negative electrode, and accelerated to the positive electrode. Each electron that is ejected induces a current pulse, the value of whose integral over the duration of the pulse \( \tau \) is the charge of an electron, \( e \). Strong illumination will cause many events during \( \tau \), resulting in an instantaneous current \( I(t) \) which fluctuates about an average \( I_{av} \). These fluctuations are what Schottky named "shot noise."

In this experiment, we wish to measure \( e \) by measuring the shot noise. We therefore need to know the dependence of the fluctuations on \( e \). Within the frequency range \( 0 < f < \frac{1}{2\pi \tau} \), the time average of the square of the current can be shown to be

\[
d < I^2 > = 2eI_{av} df.
\]

Suppose one measures the voltage across a resistor of resistance \( R_P \) where the current is \( I_{av} \). This voltage can also be approximated as a sum of Fourier components, so that its square is equal to the square of its components. With a gain of \( g(f) \), the voltage out of the circuit \( V_0 \) will be related to \( e \) as follows:

\[
V_0^2 = 2eI_{av}R_P \int_0^\infty |g(f)|^2 df + V_A^2.
\]

Here, \( V_A^2 \) is an added constant to take into account noise from other sources, specifically the amplifier and Johnson noise. It is precisely the relation in Equation 11 that will enable us to determine \( e \).

3. EXPERIMENT

3.1. Johnson Noise

Figure 1 shows a diagram of the experimental apparatus used for measurement of Johnson noise. It was first necessary to calibrate the frequency dependence of the gain from the amplifier-filter combination. To do this, we sent a sinusoidal signal from a function generator through an attenuator to the amplifier and filter. We then used an oscilloscope to record the RMS voltage \( V_J \) the input to the amplifier and \( V_0 \) at the output of the filter. We did this over a range of frequencies from 0.5 kHz to 100 kHz to find the gain of the system, \( g(f) = V_0/V_J \).

We next wanted to measure \( V_J^2 \), the mean square Johnson noise, as a function of resistance. To do this, we measured the RMS voltage at the oscilloscope with the shorting switch SW1 both open and closed for each resistance. During these measurements, SW2 was kept open, and was closed only to measure each resistance used. This procedure was used to separate the Johnson noise from the noise generated in the amplifier, so that if \( V_R \) and \( V_S \) are the RMS voltages measured when SW1 is open and closed, respectively, then

\[
V_J^2 = V_R^2 - V_S^2
\]

since these statistically independent contributions to the RMS voltage add in quadrature.

Lastly, measurements were made of the temperature dependence of Johnson noise. This was done using a single 100 k\( \Omega \) resistor. We measured \( V_R \) and \( V_S \) as above with the box with the mounted resistor inverted into a heated oven at temperatures ranging from room temperature to 150°C. We then inverted the box into a dewar of liquid nitrogen, and measured \( V_R \) and \( V_S \) at temperatures down to \(-176°C\).
3.2. Shot Noise

Figure 2 is a diagram of the apparatus used to measure the shot noise, with Figure 3 as the diagram of the photodiode and preamplifier circuit inside the photodiode box. As in the Johnson noise experiment, it was first necessary to calibrate the gain. We did this by feeding the signal from a function generator into the “test-in” input of the photodiode box, and then recording the RMS voltages $V_1$ and $V_0$ at the input of the photodiode box and at the output of the filter, respectively.

Next, to determine the shot noise, we measured the voltage $RFi$ at the “first stage output” of the photodiode box, as well as the voltage $V_0$ out of the box at the “second stage output.” Repeating this measurement over the range of intensities of the light bulb gave us the desired data in order to compute $e$.

![FIG. 2: Schematic diagram of the shot noise measurement apparatus](image)

With time, we stopped the oscilloscope and measured each voltage several times for each resistor, and then used the mean as our values for $V_R$ and $V_S$. The error in $V_f^2$ is therefore due to the propagation of the error in the means representing $V_R$ and $V_S$ [3]. We also numerically integrated to find $G$ for each different resistance used.

Since, by Equation 8, $k = V_f^2/4RT$, we were able to take the weighted mean of a measurement of $k$ for each resistance used. This gave the value $k = (2.55 \pm 0.03) \times 10^{-23}$, about twice the actual value of $k$, which is $1.38065 \times 10^{-23} \text{ J K}^{-1}$. The sources of error in this portion of the experiment are similar to those in the remaining portions, so we wait to discuss them until we have presented all of the data.

4. DATA AND ANALYSIS

4.1. Johnson Noise – Resistance Dependence

Our raw data consisted of measurements of $V_f^2$ for a variety of resistors. Since $V_R$ and $V_S$ were fluctuating with time, we stopped the oscilloscope and measured each voltage several times for each resistor, and then used the mean as our values for $V_R$ and $V_S$. The error in $V_f^2$ is therefore due to the propagation of the error in the means representing $V_R$ and $V_S$ [3]. We also numerically integrated to find $G$ for each different resistance used.

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4.2. Johnson Noise – Temperature Dependence

The analysis here follows that from the resistance dependence portion of the experiment. We had a collection of values of $V_f^2$ at different temperatures $T$. Here we only had to determine $G$ for the single resistance used, 100 kΩ. We then used Matlab to fit $V_f^2/4RT$ as a linear function $T$, which by Equation 8 should have a slope of $k$, and a $y$-intercept of absolute zero. Our fit had a slope $m = (1.03 \pm 0.08) \times 10^{-23}$ J K$^{-1}$, whereas the accepted value for Boltzmann’s constant is $k = 1.381 \times 10^{-23}$ J K$^{-1}$. Our value of $k$ differs from the accepted value by about 4%. The $y$-intercept of the linear fit is $310 \pm 30°C$. Absolute zero is actually $-273.15°C$, which is within 1.3σ of our result. The error in our results stems again from the statistical error in the measurements of $V_R$ and $V_S$. Imprecision in the determination of $G$ most likely contributed most to the discrepancies between our results and accepted values. Also, for the fit, $\chi^2 = 35$. From the graph of the fit, as seen in Figure 4, the variation in the data is noticeable. Some of it can be accounted for by the difficulty in making several RMS voltage measurements at a single temperature, since it tended to rise and fall quickly. This error would not have changed our data points, but rather could have increased the size of our error bars and thereby reducing the $\chi^2$ of the fit.

4.3. Shot Noise

We first numerically calculated the integral of the square of the gain. Then, Matlab was again employed
setup we used in our experiment leads us to believe that we incorrectly terminated the signal to the oscilloscope. Upon comparing a signal sent through the amplifier to the oscilloscope when it was terminated and not terminated, we found that the termination cut the observed RMS voltage roughly in half. This can be explained by the schematic diagram in Figure 6. If we then assume we were only observing half of the actual voltage $V_o$, we can multiply $V_o^2$ by 4 and redo the fit. This yields a linear fit, again with $\chi^2 = 1.1$, but this time with slope $(2.36 \pm 0.05) \times 10^{-19}$ C. This is 1.5 times the actual value for $e$, a much better result.

**FIG. 6:** Diagram of oscilloscope when terminated by a resistance $R_3$, where $R_1$ represents the effective resistance of the previous circuit elements. Here the oscilloscope observed a voltage $R_3 V/(R_1 + R_2)$, whereas without termination it observes a voltage $V$.

5. **CONCLUSIONS**

We identify the calculation of the integral of the gain as the primary reason for the discrepancies between our results and the accepted values. For the resistance dependence of Johnson noise, we found $k$ to be about twice its accepted value, and the different measurements we made of $k$ increased as resistances increased, even though there should be any correlation there, indicating some systematic error. The results from the temperature dependence were closer to actual values, and larger error bars to account for fluctuating temperatures could help fix this further.

The good value for $\chi^2$ in the fit for the shot noise data shows that our experimental data for the light intensity dependence was indeed very nicely linear, even if the slope is not the desired value. Another chance to perform this experiment could confirm the improper termination as the key to our initial error, and perhaps give a more precise determination of $e$.

Although our results were not accurate as we would have wished, we would like to note that this is a refreshing view of “noise.” In this experiment, rather than noise being something one wants to avoid or compensate for, it becomes a desirable quantity with which one can measure fundamental constants and properties of circuits.