In Class W10D1_3 Solutions: Coaxial Cable

**Problem:** For the coaxial cable at left (inner radius a, outer radius b):
1) How much energy is stored per unit length?
2) What is inductance per unit length?

**Solution:**
There are several ways to find energy. One is to find the inductance and then use \( U = \frac{1}{2} LI^2 \). However, since they ask us to find the inductance after finding the energy, this is unlikely to be the way to approach this problem. Another way is to consider that the energy is stored in the magnetic field, and hence find the magnetic field then integrate the energy density to find the total energy. We take this approach.

To find the field use Ampere’s law. Outside of \( b \) and inside of \( a \) the fields will be zero (because the contained current will be zero). Using the Amperian loop pictured (radius \( r \)), we find that in between the two current shells: \( \oint \mathbf{B} \cdot d\mathbf{s} = B2\pi r = \mu_0 I_{enc} = \mu_0 I \rightarrow B = \frac{\mu_0 I}{2\pi r} \) (CCW, as pictured)

The energy density is then given by: \( u_B = \frac{B^2}{2\mu_0} = \frac{1}{2\mu_0} \left( \frac{\mu_0 I}{2\pi r} \right)^2 = \frac{\mu_0 I^2}{8\pi^2 r^2} \)

Now we just need to integrate this energy density over the volume of space where we found there to be a magnetic field – in between the two shells. This is a volume integral (since \( u_B \) is an energy per unit volume), which we will do by integrating over cylindrical shells of radius \( r \) and length \( l \). We can do this because the field and hence the energy density will be constant on these shells. Also, the length is arbitrary, because we are asked to find the energy per unit length. So:

\[
U_B = \iiint u_B \, dV = \int_a^b \frac{\mu_0 I^2}{8\pi^2 r^2} \cdot 2\pi rdl = \frac{\mu_0 I^2}{4\pi} \int_a^b \frac{1}{r} dr = \frac{\mu_0 I^2}{4\pi} \ln \left( \frac{b}{a} \right)
\]

This gives us energy per unit length of: (1) \( U_{B, \text{per length}} = \frac{U_B}{l} = \frac{\mu_0 I^2}{4\pi} \ln \left( \frac{b}{a} \right) \)

To find the inductance (per unit length) we simply use the equation that relates energy and inductance: \( U = \frac{1}{2} LI^2 \), except that in this case it is actually energy per unit length on the left and inductance per unit length on the right. So

\[
U_B = \frac{1}{2} LI^2 \rightarrow I_{\text{per length}} = \frac{2U_{B, \text{per length}}}{I^2} = \frac{\mu_0}{2\pi} \ln \left( \frac{b}{a} \right)
\]